9-5 Base *e* and Natural Logarithms

Main Ideas

- Evaluate expressions involving the natural base and natural logarithms.
- Solve exponential equations and inequalities using natural logarithms.

New Vocabulary

natural base, *e*

natural base exponential function

natural logarithm

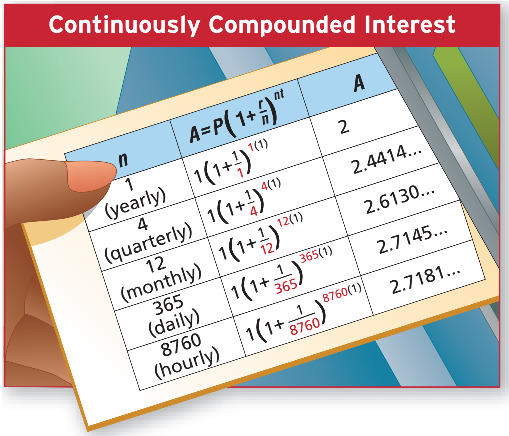
natural logarithmic function

GET READY for the Lesson

Suppose a bank compounds interest on accounts *continuously*, that is, with no waiting time between interest payments.

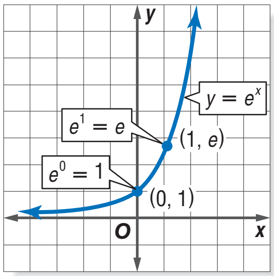
To develop an equation to determine continuously compounded interest, examine what happens to the value A of an account for increasingly larger numbers of compounding periods n. Use a principal P of \$1, an interest rate r of 100% or 1, and time t of 1 year.

Continuously Compounded Interest



Base *e* and Natural Logarithms In the table above, as n increases, the expression $1\left(1+\frac{1}{n}\right)^{n(1)}$ or $\left(1+\frac{1}{n}\right)^{n}$ approaches the irrational number 2.71828…. This number is referred to as the **natural base, *e*.**

An exponential function with base e is called a **natural base exponential function**. The graph of $y = e^x$ is shown at the right. Natural base exponential functions are used extensively in science to model quantities that grow and decay continuously.



Most calculators have an e^x function for evaluating natural base expressions.

Study Tip

Simplifying Expressions with e

You can simplify expressions involving *e* in the same manner in which you simplify expressions involving π.

Examples:

- $\pi^2 \cdot \pi^3 = \pi^5$
- $e^2 \cdot e^3 = e^5$

EXAMPLE 1 Evaluate Natural Base Expressions

Use a calculator to evaluate each expression to four decimal places.

a. e^2 **KEYSTROKES:** [2nd] [e^x] 2 [ENTER] 7.389056099

$e^2 \approx 7.3891$

b. $e^{-1.3}$ **KEYSTROKES:** [2nd] [e^x] -1.3 [ENTER] .272531793

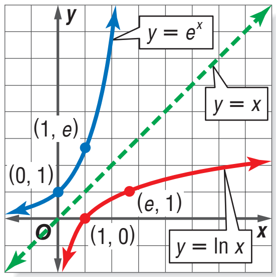
$e^{-1.3} \approx 0.2725$

CHECK Your Progress

1A. e^5

1B. $e^{-2.2}$

The logarithm with base e is called the **natural logarithm**, sometimes denoted by $\log_e x$, but more often abbreviated $\ln x$. The **natural logarithmic function**, $y = \ln x$, is the inverse of the natural base exponential function, $y = e^x$. The graph of these two functions shows that $\ln 1 = 0$ and $\ln e = 1$.



Study Tip

Calculator Keystrokes
On graphing calculators, you press the LN key before the number. On other calculators, usually you must type the number before pressing the LN key.

Most calculators have an LN key for evaluating natural logarithms.

EXAMPLE Evaluate Natural Logarithmic Expressions

2 **Use a calculator to evaluate each expression to four decimal places.**

a. $\ln 4$ **KEYSTROKES:** LN 4 ENTER 1.386294361

$\ln 4 \approx 1.3863$

b. $\ln 0.05$ **KEYSTROKES:** LN 0.05 ENTER –2.995732274

$\ln 0.05 \approx -2.9957$

CHECK Your Progress

2A. $\ln 7$ **2B.** $\ln 0.25$

You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that $\ln x = \log_e x$.

EXAMPLE Write Equivalent Expressions

3 **Write an equivalent exponential or logarithmic equation.**

a. $e^x = 5$

$e^x = 5 \longrightarrow \log_e 5 = x$

$\ln 5 = x$

b. $\ln x \approx 0.6931$

$\ln x \approx 0.6931 \longrightarrow \log_e x \approx 0.6931$

$x \approx e^{0.6931}$

CHECK Your Progress

3A. $e^x = 6$ **3B.** $\ln x \approx 0.5352$

Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to "undo" each other.

$$e^{\ln x} = x \qquad \ln e^x = x$$

For example, $e^{\ln 7} = 7$ and $\ln e^{4x + 3} = 4x + 3$.

Equations and Inequalities with *e* and ln Equations and inequalities involving base e are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

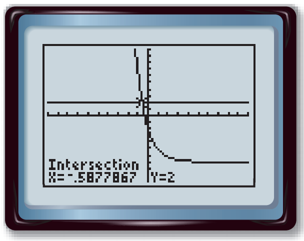
EXAMPLE 4 Solve Base e Equations

Solve $5e^{-x} - 7 = 2$. Round to the nearest ten-thousandth.

$5e^{-x} - 7 = 2$	Original equation
$5e^{-x} = 9$	Add 7 to each side.
$e^{-x} = \frac{9}{5}$	Divide each side by 5.
$\ln e^{-x} = \ln \frac{9}{5}$	Property of Equality for Logarithms
$-x = \ln \frac{9}{5}$	Inverse Property of Exponents and Logarithms
$x = -\ln \frac{9}{5}$	Divide each side by −1.
$x \approx -0.5878$	Use a calculator.

The solution is about −0.5878.

CHECK You can check this value by substituting −0.5878 into the original equation and evaluating, or by finding the intersection of the graphs of $y = 5e^{-x} - 7$ and $y = 2$.



CHECK Your Progress

Solve each equation. Round to the nearest ten-thousandth.

4A. $3e^{x} + 2 = 4$ **4B.** $4e^{-x} - 9 = -2$

Study Tip

Continuously Compounded Interest

Although no banks actually pay interest compounded continuously, the equation $A = Pe^{rt}$ is so accurate in computing the amount of money for quarterly compounding, or daily compounding, that it is often used for this purpose.

When interest is compounded continuously, the amount A in an account after t years is found using the formula $A = Pe^{rt}$, where P is the amount of principal and r is the annual interest rate.

Real-World EXAMPLE 5 Solve Base e Inequalities

SAVINGS **Suppose you deposit \$1000 in an account paying 2.5% annual interest, compounded continuously.**

a. What is the balance after 10 years?

$A = Pe^{rt}$	Continuous compounding formula
$= 1000e^{(0.025)(10)}$	Replace P with 1000, r with 0.025, and t with 10.
$= 1000e^{0.25}$	Simplify.
≈ 1284.03	Use a calculator.

The balance after 10 years would be \$1284.03.

CHECK If the account was earning simple interest, the formula for the interest, would be $I = prt$. In that case, the interest would be $I = (1000)(0.025)(10)$ or \$250. Continuously compounded interest should be greater than simple interest at the same rate. Thus, the solution \$1284.03 is reasonable.

b. **How long will it take for the balance in your account to reach at least $1500?**

Words	The balance is at least $1500.
Variable	Let A represent the amount in the account.
Inequality	$A \geq 1500$

$\ln e^{(0.025)t} \geq 1500$	Replace A with $1000e^{(0.025)t}$.
$\ln e^{(0.025)t} \geq 1.5$	Divide each side by 1000.
$\ln e^{(0.025)t} \geq \ln 1.5$	Property of Equality for Logarithms
$0.025t \geq \ln 1.5$	Inverse Property of Exponents and Logarithms
$t \geq \frac{\ln 1.5}{0.025}$	Divide each side by 0.025.
$t \geq 16.22$	Use a calculator.

It will take at least 16.22 years for the balance to reach $1500.

CHECK Your Progress

Suppose you deposit $5000 in an account paying 3% annual interest, compounded continuously.

5A. What is the balance after 5 years?

5B. How long will it take for the balance in your account to reach at least $7000?

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EXAMPLE Solve Natural Log Equations and Inequalities

6 **Solve each equation or inequality. Round to the nearest ten-thousandth.**

a. $\ln 5x = 4$

$\ln 5x = 4$	Original equation
$e^{\ln 5x} = e^4$	Write each side using exponents and base *e*.
$5x = e^4$	Inverse Property of Exponents and Logarithms
$x = \frac{e^4}{5}$	Divide each side by 5.
$x \approx 10.9196$	Use a calculator. Check using substitution or graphing.

b. $\ln (x - 1) > -2$

$\ln (x - 1) > -2$	Original inequality
$e^{\ln (x-1)} > e^{-2}$	Write each side using exponents and base *e*.
$x - 1 > e^{-2}$	Inverse Property of Exponents and Logarithms
$x > e^{-2} + 1$	Add 1 to each side.
$x > 1.1353$	Use a calculator. Check using substitution.

CHECK Your Progress

6A. $\ln 3x = 7$

6B. $\ln (3x + 2) < 5$

Study Tip

Equations with ln

As with other logarithmic equations, remember to check for extraneous solutions.

Check Your Understanding

Examples 1, 2 (pp. 536, 537)

Use a calculator to evaluate each expression to four decimal places.

1. e^6 **2.** $e^{-3.4}$ **3.** $e^{0.35}$

4. ln 1.2 **5.** ln 0.1 **6.** ln 3.25

Example 3 (p. 537)

Write an equivalent exponential or logarithmic equation.

7. $e^x = 4$ **8.** $\ln 1 = 0$

Example 4 (p. 538)

Solve each equation. Round to the nearest ten-thousandth.

9. $2e^x - 5 = 1$ **10.** $3 + e^{-2x} = 8$

Example 5 (pp. 538–539)

ALTITUDE For Exercises 11 and 12, use the following information.
The altimeter in an airplane gives the altitude or height h (in feet) of a plane above sea level by measuring the outside air pressure P (in kilopascals).

The height and air pressure are related by the model $P = 101.3\, e^{-\frac{h}{26{,}200}}$.

11. Find a formula for the height in terms of the outside air pressure.

12. Use the formula you found in Exercise 11 to approximate the height of a plane above sea level when the outside air pressure is 57 kilopascals.

Example 6 (p. 539)

Solve each equation or inequality. Round to the nearest ten-thousandth.

13. $e^x > 30$ **14.** $\ln x < 6$

15. $2 \ln 3x + 1 = 5$ **16.** $\ln x^2 = 9$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
17–20	1
21–24	2
25–32	3
33–40	4
41–46	5
47–54	6

Use a calculator to evaluate each expression to four decimal places.

17. e^4 **18.** e^5 **19.** $e^{-1.2}$ **20.** $e^{0.5}$

21. ln 3 **22.** ln 10 **23.** ln 5.42 **24.** ln 0.03

Write an equivalent exponential or logarithmic equation.

25. $e^{-x} = 5$ **26.** $e^2 = 6x$ **27.** $\ln e = 1$ **28.** $\ln 5.2 = x$

29. $e^{x+1} = 9$ **30.** $e^{-1} = x^2$ **31.** $\ln \frac{7}{3} = 2x$ **32.** $\ln e^x = 3$

Solve each equation. Round to the nearest ten-thousandth.

33. $3e^x + 1 = 5$ **34.** $2e^x - 1 = 0$ **35.** $-3e^{4x} + 11 = 2$ **36.** $8 + 3e^{3x} = 26$

37. $2e^x - 3 = -1$ **38.** $-2e^x + 3 = 0$ **39.** $-2 + 3e^{3x} = 7$ **40.** $1 - \frac{1}{3}e^{5x} = -5$

POPULATION For Exercises 41 and 42, use the following information.
In 2005, the world's population was about 6.5 billion. If the world's population continues to grow at a constant rate, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2005.

41. According to this model, what will the world's population be in 2015?

42. Some experts have estimated that the world's food supply can support a population of at most 18 billion. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

Real-World Link

To determine the doubling time on an account paying an interest rate r that is compounded annually, investors use the "Rule of 72." Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $\frac{72}{6}$ or 12 years.

Source: datachimp.com

MONEY **For Exercises 43–46, use the formula for continuously compounded interest found in Example 5.**

43. If you deposit \$100 in an account paying 3.5% interest compounded continuously, how long will it take for your money to double?

44. Suppose you deposit A dollars in an account paying an interest rate of r, compounded continuously. Write an equation giving the time t needed for your money to double, or the *doubling time*.

45. Explain why the equation you found in Exercise 44 might be referred to as the "Rule of 70."

46. **MAKE A CONJECTURE** State a rule that could be used to approximate the amount of time t needed to triple the amount of money in a savings account paying r percent interest compounded continuously.

Solve each equation or inequality. Round to the nearest ten-thousandth.

47. $\ln 2x = 4$ **48.** $\ln 3x = 5$ **49.** $\ln (x + 1) = 1$ **50.** $\ln (x - 7) = 2$

51. $e^x < 4.5$ **52.** $e^x > 1.6$ **53.** $e^{5x} \geq 25$ **54.** $e^{-2x} \leq 7$

E-MAIL **For Exercises 55 and 56, use the following information.**

The number of people N who will receive a forwarded e-mail can be approximated by $N = \frac{P}{1 + (P - S)e^{-0.35t}}$, where P is the total number of people online, S is the number of people who start the e-mail, and t is the time in minutes. Suppose four people want to send an e-mail to all those who are online at that time.

55. If there are 156,000 people online, how many people will have received the e-mail after 25 minutes?

56. How much time will pass before half of the people will receive the e-mail?

EXTRA PRACTICE
See pages 911, 934.
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Solve each equation. Round to the nearest ten-thousandth.

57. $\ln x + \ln 3x = 12$ **58.** $\ln 4x + \ln x = 9$

59. $\ln (x^2 + 12) = \ln x + \ln 8$ **60.** $\ln x + \ln (x + 4) = \ln 5$

H.O.T. Problems

61. **OPEN ENDED** Give an example of an exponential equation that requires using natural logarithms instead of common logarithms to solve.

62. **FIND THE ERROR** Colby and Elsu are solving $\ln 4x = 5$. Who is correct? Explain your reasoning.

Colby	Elsu
$\ln 4x = 5$	$\ln 4x = 5$
$10^{\ln 4x} = 10^5$	$e^{\ln 4x} = e^5$
$4x = 100{,}000$	$4x = e^5$
$x = 25{,}000$	$x = \frac{e^5}{4}$
	x 37.1033

63. **CHALLENGE** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For all positive numbers x and y, $\frac{\log x}{\log y} = \frac{\ln x}{\ln y}$.

64. *Writing in Math* Use the information about banking on page 536 to explain how the natural base e is used in banking. Include an explanation of how to calculate the value of an account whose interest is compounded continuously, and an explanation of how to use natural logarithms to find the time at which the account will have a specified value in your answer.

STANDARDIZED TEST PRACTICE

65. **ACT/SAT** A recent study showed that the number of Australian homes with a computer doubles every 8 months. Assuming that the number is increasing continuously, at approximately what monthly rate must the number of Australian computer owners be increasing for this to be true?

A 68%

B 8.66%

C 0.0866%

D 0.002%

66. **REVIEW** Which is the first *incorrect* step in simplifying $\log_3 \frac{3}{48}$?

Step 1: $\log_3 \frac{3}{48} = \log_3 3 - \log_3 48$

Step 2: $= 1 - 16$

Step 3: $= -15$

F Step 1

G Step 2

H Step 3

J Each step is correct.

Spiral Review

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

67. $\log_4 68$

68. $\log_6 0.047$

69. $\log_{50} 23$

Solve each equation. Check your solutions. (Lesson 9-3)

70. $\log_3 (a + 3) + \log_3 (a - 3) = \log_3 16$

71. $\log_{11} 2 + 2 \log_{11} x = \log_{11} 32$

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation. (Lesson 8-4)

72. $mn = 4$

73. $\frac{a}{b} = c$

74. $y = -7x$

75. **BASKETBALL** Alexis has never scored a 3-point field goal, but she has scored a total of 59 points so far this season. She has made a total of 42 shots including free throws and 2-point field goals. How many free throws and 2-point field goals has Alexis scored? (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest hundredth.** (Lesson 9-1)

76. $2^x = 10$

77. $5^x = 12$

78. $6^x = 13$

79. $2(1 + 0.1)^x = 50$

80. $10(1 + 0.25)^x = 200$

81. $400(1 - 0.2)^x = 50$